Team Control Number

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Problem Chosen

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Summary Sheet

Modeling the Spread of Dandelion Using Coupled PDE

The dandelion's success as a widespread perennial largely lies in the parachute-like structure of its seeds, which allows them to be dispersed by the wind. To model the effect of wind and other factors including temperature, moisture, climate, species competitions, etc. on dandelion's population, we used **coupled partial differential equations** (PDE) in our mathematical model.

In this paper, we constructed a **Dandelion Spread PDE Model** (DSM) to investigate the spread of dandelions over time. Our model is capable of accurately describing the dandelion population at every life stage, each moment in time, and any coordinate in the one-hectare plot of land. The DSM model is composed of four PDEs. We first developed the PDE for settled dandelion seed population density by subtracting the number of seeds that germinate from the number of seeds that fall from the air. The PDE for dandelion plants' population densities incorporates death rate and germination rate. Temperature and humidity's effects on both rates were considered. Furthermore, referring to the **Fisher model**, we corrected the equation by multiplying a logistic term to obtain a logistic population growth that depicts the effect of **intraspecific competition** on the dandelion population. The PDE for dandelion puffballs directly affects the number of seeds being dispersed in the air. We used **advection-diffusion equation** to model such drifting dandelion seeds by adding **Brownian Random Dispersal** to the equation. The advection-diffusion equation allows us to effectively predict the spread of dandelion in various kinds of winds.

Then, we collected data and carefully determined the values of the parameters through empirical analysis and data fitting. We transform the **strong form PDEs** into **weak forms** and solve them using the method of **finite element**. We used FEniCS to obtain the final predictions of the DSM model. The results were analyzed and a sensitivity analysis was conducted.

To measure the dandelion's invasiveness, we modelled its impacts on native species. We modified DSM by subtracting the **interspecific competition** correction term. We also utilized a modified Fisher Equation to model native plants' population densities. Specifically, we considered three indicators to evaluate the invasiveness of a species: change in **total biomass** of native species, its economic value, and its impacts on the environment. The total biomass is calculated using a modified Yield and Density Equation. We used **Analytic Hierarchy Process (AHP)** to determine the weights of indicators, and sum up the products together to obtain the final "impact factor". For the calculation of the two other invasive species' population densities in the computation of their "impact factors", we utilized the same modified Fisher Equation we used for native species.

Keywords: **Coupled Partial Differential Equation, Dandelion Spread PDE Model, Modified Fisher Model, Interspecific and Intraspecific Competition, Brownian Random Dispersal, Advection-diffusion Equation, Biomass, Modified Yield and Density Equation, Analytic Hierarchy Process**

Contents

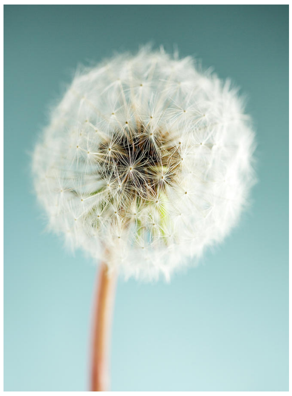
1 Introduction

1.1 Background

Taraxaxum officinale, commonly known as the dandelion, is a widespread perennial that can live up to 10-13 years under undisturbed conditions. There are many factors that would affect the growth of dandelion, including temperature, moisture, and wind speed. The dandelion's life cycle typically consists of four stages: seed, plant, flower, and puffball. The finish of seed dispersion marks the end of the dandelion's puffball stage, and it will grow back into a plant. The dandelion is most commonly recognized in its puffball stage, during which seeds are allowed to be dispersed by wind with the pappus facing out[1]. The parachute shape of dandelion seeds allows dandelions to spread to places far from their origin, and after landing seeds may form a persistent seedbank under the ground, normally with a half-life of 3 months[2].

Dandelions are so widespread that they may be found worldwide and in almost all types of disturbed areas[3]. They are considered to be invasive species because they can have rapid reproduction and strong adaptability to various conditions, and may compete resources with native species. On the other hand, dandelions hold economic value. Dandelions are used as medicine for stomach, liver, and kidney disorders. This is because the dandelion contains a high concentration of vitamins and minerals. People also use dandelions to make food and drink such as dandelion tea and seasoning[4].

(a) Dandelion Field[24]



(b) Dandelion Puffball[25]

1.2 Problem Restatement

- We are asked to model the spread of dandelion over 1,2,3,6,12 months in an open one-hectare land, given that initially a single dandelion puffball is adjacent to the land. We are also asked to consider various factors, including temperature, humidity, and climate, which impact dandelion growth.
- We should develop a method to determine "impact factors" for invasive species, considering variables such as the species' economic and environmental harm, adaptability to new environments, the condition of the environment, and competition with native species.
- We need to examine our model by computing a factor for dandelion. The factor should be able to reflect the extent of dandelion's harm to the local habitat after careful determinations of dandelion's related variables.
- We also should choose two other species and compute the "impact factors" for both of

them. It's crucial to identify the region being considered, as the condition of that specific region directly influences the classification of the species as invasive.

2 Assumptions and Justifications

- We assume that a dandelion will not return to the plant stage after the puffball stage is finished and will be cleared from the system. This will greatly simplify the complexity of the program, and since we are only simulating population change within one year, it won't have a significant impact. Also, the number of seeds produced by a plant is negligible compared to the total seeds it produces, and thus will not affect the final outcome.
- Puffballs have a constant seed production rate every day. This is to simplify the model when modeling the production of seeds.
- The falling rate of drifting seeds in the air is a constant for simplicity of the model and for modeling the shifting of drifting seeds to settled seeds.
- Germination rate and death rates are only determined by temperature and moisture. According to relevant literature, these are the two most significant factors[10].
- We also assume that the time a dandelion remains in one stage is constant for the sake of simplifying the model.

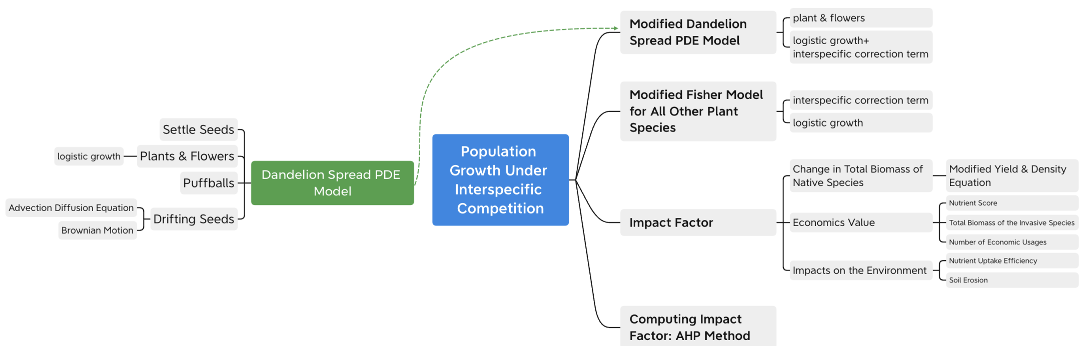
3 The development of Models

3.1 Variables

Symbol	Description	Unit
$u_s(x,y,t)$	The mean population density of settled dandelion seeds	m^{-2}
$u_p(x,y,t)$	The mean population density of dandelion plants	m^{-2}
$u_{puf}(x,y,t)$	The mean population density of dandelion puffballs	m^{-2}
$u_d(x,y,t)$	The mean population density of drifting dandelion seeds	m^{-2}
n_s	The average total time required for a dandelion seed to germinate	day
n_p	The average total time required for a dandelion plant to mature	day
n_{puf}	The average total time required for a dandelion puffball to seed	day
ψ	The falling rate of a drifting dandelion seed	N/A
g	The germination rate of a settled dandelion seed	N/A
λ	The average number of seeds a mature dandelion puffball produces	N/A
$\vec{v}$	The speed vector of dandelion	
δ_p	The death rate of dandelion at plant and flower stage	m
δ_{puf}	The death rate of dandelion at puff stage	N/A
ΔW_T	Change in total biomass of native species	g
Θ	The economic value of a given invasive plant species	N/A
Φ	Impacts of a given invasive species on the environment	N/A
I	The final calculated impact factor of dandelion	N/A

3.2 Model Overview

Below is the overview of our model. We developed four models in total: Dandelion Spread PDE Model, Modified Dandelion Spread Model Under Competition, Modified Fisher Model for all other plant species, and the final AHP evaluation model for computation of the "impact factor".



4 Dandelion Spread PDE Model (DSM)

In this problem, we construct a series of partial differential equations (PDE) to model the dispersion of dandelions over time. Our dandelion spread PDE model (DSM) takes into account the influence of multiple factors including arid, climates, temperate, etc., and is divided into four sections corresponding to the development of a dandelion in different stages of its lifespan. Combining this model with relevant programming tools such as FEniCS, we are able to model and predict the spread of dandelions adjacent to an open one-hectare land over several months.

4.1 Establishing a Temporal Variation of Population Density Using PDE

To analyze the dispersion of dandelions, we categorize their growth into four stages: settled seeds, dandelion plants (including the flowering phase, as dandelion flowers do not impact seed dispersion), dandelion puffballs, and drifting seeds in the air. The process is straightforward: a settled seed grows into a plant, flowers, and becomes a puffball. The puffball then releases its seeds into the air and the airborne seeds would drift and land randomly on new territories, becoming settled seeds again. It is worth mentioning that although puffballs after seeding would again become plants, compared to the number of seeds it has released, the increase in plant population is relatively minute and can be ignored since it generates redundant calculations and uncertainties.

In order to analyze such thorough dispersion processes of dandelion populations, our model examines the temporal variation of the dandelion population density u that is relevant to the object's x-axis coordinate x, y-axis coordinate y, and time t for each of the stages by implementing partial differential equations.

4.1.1 PDE for Settled Dandelion Seeds

To begin with, for settled seeds that are buried in the dirt, the temporal variations of seed population density are calculated via the source from airborne, dropping seeds, and the loss of settled seeds that are mature enough to grow up to the plant stage, as shown in Formula 4.1.

$$\frac{\partial u_s}{\partial t} = \psi \cdot u_d - \psi \cdot u_d(x, y, t - n_s) \tag{4.1}$$

In which u_s represents the population density of settled seeds while u_d denotes the population density of drifting seeds before landing. n_s is the total time required for a settled seed to become a plant starting from the instant the dandelion lands on the ground from the airborne state. t shows the time in days. ψ denotes the falling rate of a drifting seed so that the first term $\psi \cdot u_d$ stands for the number of airborne seeds per unit area (u_d) fallen onto the ground, serving as the major source. On the other hand, $\psi \cdot u_d(x, y, t - n_s)$ identifies the loss of seeds due to germination (they are now plants instead) since they landed and settled n_s days ago, which is the exact time required to germinate.

4.1.2 PDE for Dandelion Plants

A dandelion then undergoes development into its plant stage, when it grows and flowers. Because a dandelion plant and a dandelion flower share the same living pattern and flowering does not affect seed dispersion, our definition of a dandelion plant encompasses both stages. Also, as justified before, the increase in plant density due to puffballs that have seeded is ignored in this section due to its minor influence and redundant calculations. Additionally, although intraspecific competition exists during dandelion dispersion, we do not take into account the interspecific effects from other species. Equations considering the influence of interspecific competition will be covered in Section 5.

In this way, the time variation of the density of a dandelion plant population (noted as u_p) will depend mainly on germinated seeds and the loss of plants due to mortality and maturation. We also need to take into account the impact of intraspecific competitions while calculating the matured density of dandelions (other situations like mortality do not involve intraspecific competitions).

While the calculation of mature dandelion population density becomes more complex, we now define a new function in order to simplify further notations. It is worth mentioning that since we define n as the total time required for a dandelion to become mature in its stage (n_p states that a dandelion plant has to be n_p days old from being sowed to grow into a puffball), the density of dandelions that are mature under the plant stage or the puffball stage can be calculated using a function $\sigma(n)$, namely the mature dandelion population density. Note that we will only consider plants and puffballs since the last stage – the drifting seed stage – does not involve maturation.

Define $\sigma(n)$ as

$$\sigma(n) = g \cdot \psi \cdot u_d(x, y, t-n) \cdot (1 - \frac{u_p}{N}) \tag{4.2}$$

Where $\psi \cdot u_d(x, y, t-n)$ simply means that because the matured dandelion population has to be n days old, we can trace back to the population density of drifting seeds that have fallen onto the ground at time $(t-n)$. Note that n_{puf} has to be larger than n_p because a dandelion puffball has to first become a plant. Additionally, a matured dandelion needs to germinate when it was a seed, so a germination rate g is multiplied. The germination rate can be calculated using Formula 4.3 below.

$$g = g_{base} \cdot K_t(T) \cdot K_w(W) \tag{4.3}$$

Where g_{base} stands for the default condition, which is the maximum germination rate of dandelion seeds. However, the germination rate does not always equal g_{base} since it is affected by the temperature and the humidity. Therefore, $K_t(T)$ resembles the effect of changing temperatures with unit Celsius, while $K_w(W)$ indicates the effect of changing water potential W with units MPa. Both functions are regressed through relevant collected data – $K_t(T)$ fits a quadratic function, as dandelion seeds require an optimized temperature to germinate, and $K_w(W)$ fits a logistic function, referencing Luo and Cardina's 2012 work [8]. The regressed equations and graphs of the two functions are put into the Appendices.

It is worth mentioning that $K_t(T)$ and $K_w(W)$ also affect the mortality rate of dandelion plants and puffballs, namely δ_p and δpuf, as shown in Formula 4.4 and 4.5.

$$\delta_p = [1 - (1 - \delta_p^{base}) \cdot K_t(T) \cdot K_w(W)]0.00274 \tag{4.4}$$

$$\delta_{puf} = [1 - (1 - \delta_{puf}^{base}) \cdot K_t(T) \cdot K_w(W)]0.00274 \tag{4.5}$$

δ^{base} here denotes the minimum annual death rate in natural conditions, hence $(1 - \delta^{base})$ identifies the annual survival rate. By multiplying with the functions of temperature and humidity as they affect dandelion mortality and using 1 to subtract such expression, we obtain the actual annual death rate of dandelion plants and puffballs. After data analyzing and fitting we use coefficients to obtain a required death rate [23]. In default circumstances, we assume that δ_p^{base} is 0.2, δ_{puf}^{base} is 0.1, T is 20 Celsius, and W is -0.8 Mpa. Details are included in the parameter settings in 4.3.1 and justifications.

Linking back to Formula 4.2, the last term of the function illustrates the logistic population growth of dandelion plants and puffballs due to intraspecific competition, as N stands for the maximum density, referring to the Fisher model covered by Holmes, Lewis, Banks, and Veit in their 1994 work. [7]

Consequently, the PDE for the spread of dandelion plants is shown in Formula 4.6.

$$\frac{\partial u_p}{\partial t} = \sigma(n_s) - (1-\delta_p)\cdot\sigma(n_p) - \delta_p \cdot u_p \tag{4.6}$$

Where u_p denotes the mean population density of dandelion plants. The first term on the right-hand side identifies the population density of mature settled seeds, as they require n_s days to develop into plants.

The second term, similar to the first, illustrates the density of plants that are already mature since they are currently n_p days old. $(1-\delta_p)$ here represents the survival probability.

The third term stands for the mortality of dandelion plants as δ_p indicates the actual death rate, hence it needs to be subtracted.

4.1.3 PDE for Dandelion Puffballs

Next, a living dandelion moves on to its puffball phase, during which it prepares to sow. This stage is similar to the plant stage, as $\partial u_{puf}/\partial t$ depends primarily on three factors: the maturation of surviving dandelion plants, the maturation of puffballs, and the mortality of puffballs, as shown in Formula 4.7.

$$\frac{\partial u_{puf}}{\partial t} = (1-\delta_p)\cdot\sigma(n_p) - (1-\delta_{puf})\cdot(1-\delta_p)\cdot\sigma(n_{puf}) - \delta_{puf}\cdot u_{puf} \tag{4.7}$$

Where u_{puf} stands for the population density of dandelion puffballs. The first term is the same as the one in Formula 4.6, representing the mature dandelion plant population density.

Subsequently, the next term denotes the maturation of puffballs that are n_{puf} days old, multiplied by the survival rate of puffballs. However, a puffball has to first thrive as a plant, hence the survival rate of dandelion plants is multiplied.

Then the PDE subtracts the normal mortality of puffballs deduced from the death rate δ_{puf}.

4.1.4 Advection-Diffusion Equation for Drifting Dandelion Seeds (ADE)

Lastly, the spread of dandelion populations is mainly based on the dispersion of airborne seeds released from puffballs that drift and land randomly. The population density of moving seeds showcased as u_d is affected by multiple factors such as the seed production of puffballs, Brownian motion, advection by wind or other factors, and falling.

It is worth mentioning that by referring to the general advection-diffusion equation from Stocker's work in 2011 [5], we are able to modify such situations. The general advection-diffusion equation is shown in Formula 4.8.

$$\frac{\partial C}{\partial t} = -\nabla \cdot (\vec{u}C) + \nabla \cdot (D\nabla C) + P \tag{4.8}$$

Where C is the mean density, P is the net source density, and D is the diffusion coefficient, indicating that the second term illustrates the random Brownian motion. Meanwhile, ∇ serves as the divergence operator that "acts on vectors and yields the 'scalar product' of ∇ and the vector":

$$\nabla \cdot \vec{u} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \tag{4.9}$$

Our DSM model modifies such advection-diffusion equation based on the actual factors impacting moving dandelion seeds, as shown in Formula 4.10.

$$\frac{\partial u_d}{\partial t} = -\vec{v} \cdot \nabla u_d + D \cdot \nabla^2 u_d + \lambda \cdot u_{puf} - \psi \cdot u_d \tag{4.10}$$

In the modified ADE above, all the densities u are not written in $\sigma(n)$ because there is no maturation involved. The first term accounts for advection, or drifting away of airborne seeds on the x-y plane due to factors like wind. This is achieved by obtaining the dot product of the vector $\vec{v}$ that represents the drifting velocity and the divergence operator ∇, serving as the coefficient for the density u_d.

The second term is simply a reformulation of the density increase due to Brownian motion stated in Formula 4.8. It appears in many random motion (RM) models such as the one in Gurney and Nisbet's 1974 paper [6].

The third and fourth terms denote specific source density variations. As λ indicates the number of seeds released by a living puffball per day, the third term illustrates the density increase due to seeds produced. On the other hand, as mentioned previously in Formula 4.1, the fourth term stands for the density of airborne seeds fallen to the surface and should be subtracted.

4.1.5 Testing and Visualizing the Spread of Drifting Dandelion Seeds

Since ADE illustrates the most essential and evident dispersion of dandelions – the spread of moving seeds in the air – it is crucial for us to test and visualize the spread of drifting seeds and the corresponding spread of settled seeds under simplified settings.

To visualize the spread of dandelion seeds within a designated area over time, we need to transform the strong form of the our PDE equations into weak forms [9], and use FEniCS as a tool to obtain visualizations with given parameters. The details of employing FEniCS as a visualization tool will be covered in Section 4.2.

For the current test, we use simple parameters for Formula 4.10, as shown in the table.

In our simplified test, we directly denote the initial u_d as one when x, y are less than two, meaning that there are four airborne seeds initially at the corner in the given one-hectare area.

Plant	Initial Population	$\vec{v}$	D	ψ
Dandelion	4(Drifting Seeds)	$(50, 40)$	10	15

Table 4.1: Initial Parameter Settings for the Seed Dispersion Test

This avoids the calculations of matured settled seeds or drifting seeds produced by puffballs since these two terms are both zero. Next, we assume that the velocity of the wind, which is the only influencing factor in this situation, constantly remains as $(50, 40)$, indicating that the wind blows in approximately the northeastern direction. The Brownian motion and falling of the seeds can be computed using given parameters. t is set to be five days. Therefore, to see how the seeds drift and settle under the effect of the blowing wind from day 1 to day 5, using FEniCS we can generate the following graphs.

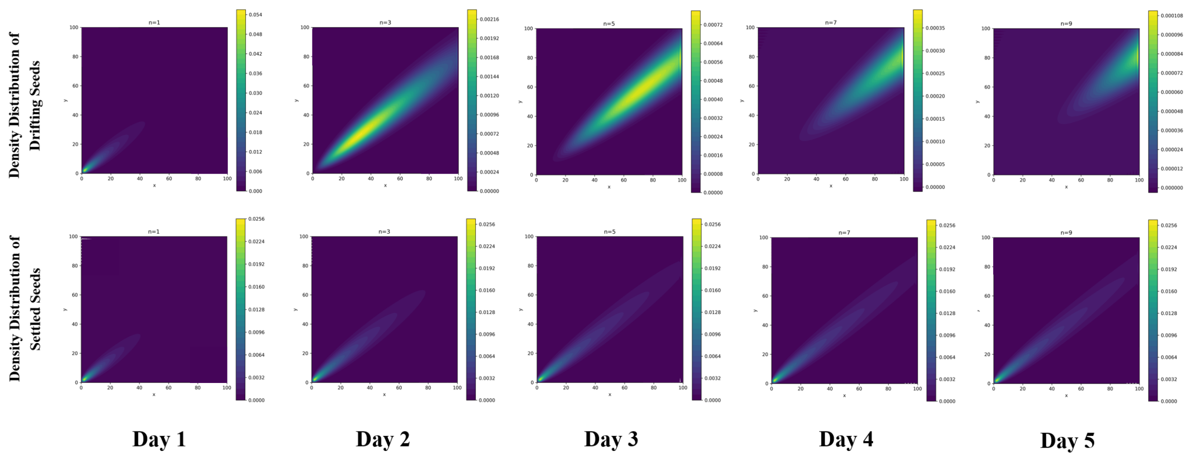


Figure 1: Density Distribution Graph of the Seed Dispersion Test

The columns are arranged from day 1 to 5. The first row outlines the density distribution of drifting seeds in the air, and because we generate two graphs of the same kind for each day, the plot number n is 1,3,5,7,9. Also note the change of the color bar – the units decrease for every graph of drifting seeds because as time proceeds, the dispersing seeds become more diverse, and some of them fall on the ground. The second row is much alike, showcasing the density distribution of settled seeds on the surface.

We can see that as the wind blows in a $(50, 40)$ velocity, the seeds diffuse exactly in the direction of the wind – from the southwest corner towards the northeast corner. As the drifting seeds spread some of them fall onto the ground due to gravity, so the maximum density point of drifting seeds shifts from the southwestern corner to somewhere near the northeastern corner while the mean density decreases (as shown in the color bar). On the other hand, the density of settled seeds increases along the past path of drifting seeds because as drifting seeds are blown by the wind they have a 15 percent probability to fall. Because settled seeds do not possess mobility, the density variation of the second row is relatively small. Hence, this is a straightforward and efficient way of visualizing the effect of wind on the advection, diffusion, and landing of dandelion seeds on a one-hectare land.

4.2 Solving Coupled PDE

The PDE model is solved using the method of finite element implemented in FEniCS. In order to perform it, we need to transform the original strong form PDE to a weak form through

3 steps. We take the modified ADE (Equation 4.10) as an example to illustrate it.

Step 1: Discretize Time

We approximate the partial derivative on the left side of Equation 4.10 as:

$$\frac{u_d^{n+1} - u_d^n}{\Delta t} = -\vec{v} \cdot \nabla u_d^{n+1} + D \cdot \nabla^2 u_d^{n+1} + \lambda \cdot u_{puf} - \psi \cdot u_d^{n+1}, \tag{4.11}$$

where the superscripts of n and $n+1$ represents the time state they are under.

Step 2: Move every term to one side of the equation and times every term by a testing function v

This results in:

$$v[u_d^{n+1} - u_d^n - \Delta t(-\vec{v} \cdot \nabla u_d^{n+1} + D \cdot \nabla^2 u_d^{n+1} + \lambda \cdot u_{puf} - \psi \cdot u_d^{n+1})] = 0, \tag{4.12}$$

where the testing function v is used to approximate u for obtaining solution.

Step 3: Integrate the function in its spatial domain Ω and use integration by part to eliminate second derivative term

This results in:

$$\int_\Omega (\Delta^{-1}(u_d^{n+1} - u_d^n)v + \vec{v} \cdot \nabla u_d^{n+1} v + D \nabla u_d^{n+1} \nabla v + \psi \cdot u_d^{n+1} v - \lambda \cdot u_{puf} v) dx = 0, \tag{4.13}$$

which can be plugged into FEniCS for solving.

4.3 Dandelion Dispersion Predictions and Analysis Using DSM and FEniCS

Based on FEniCS and our DSM model, We aim to solve the issue of predicting the spread of dandelions over the course of 1, 2, 3, 6, and 12 months if a single puffball is adjacent to an open one-hectare plot of land.

4.3.1 Initial Settings

In the given issue, one dandelion at its puffball stage is at the edge of a one-hectare plot of land. We made careful estimations of the values of the parameters based on past papers and conducted reasonable empirical analysis to determine the final values[2, 3, 8, 10, 12]. λ and g_{base} are lower than researchers normally suggest because applying the model we discovered the actual chance for a dandelion seed to land on the ground which satisfies germinate conditions is much lower than germination rates in labs.

Plant	Initial Population	$\vec{v}$	N_i	D_i	ψ	n_s	n_p	n_{puf}	λ
Dandelion	1 (puffball)	$\vec{v} \in [-10, 75]$	50	10	0.9	14	90	95	20

Table 4.2: Initial Parameter Settings(a)

Plant	g_{base}	δ_p^{base}	δ_{puf}^{base}
Dandelion	0.5	0.2	0.1

Table 4.3: Initial Parameter Settings(b)

4.3.2 Dandelion Population Density Distribution Graphs

With modified parameters according to the condition and the four main PDEs from our DSM model, we are able to generate dandelion population density distribution graphs under a certain time using FEniCS as a programming tool.

Given the fact that one puffball was adjacent to the one-hectare area at $(0, 0)$, $t = 0$, we come up with four sets of population density distribution graphs of dandelions over time: of dandelion drifting seeds, of dandelion settled seeds, of dandelion plants, and of dandelion puffballs, each containing seven values of t. To be specific, we include the distributions when t equals nine months and 11 months (assume that a month contains 30 days except when t is a year).

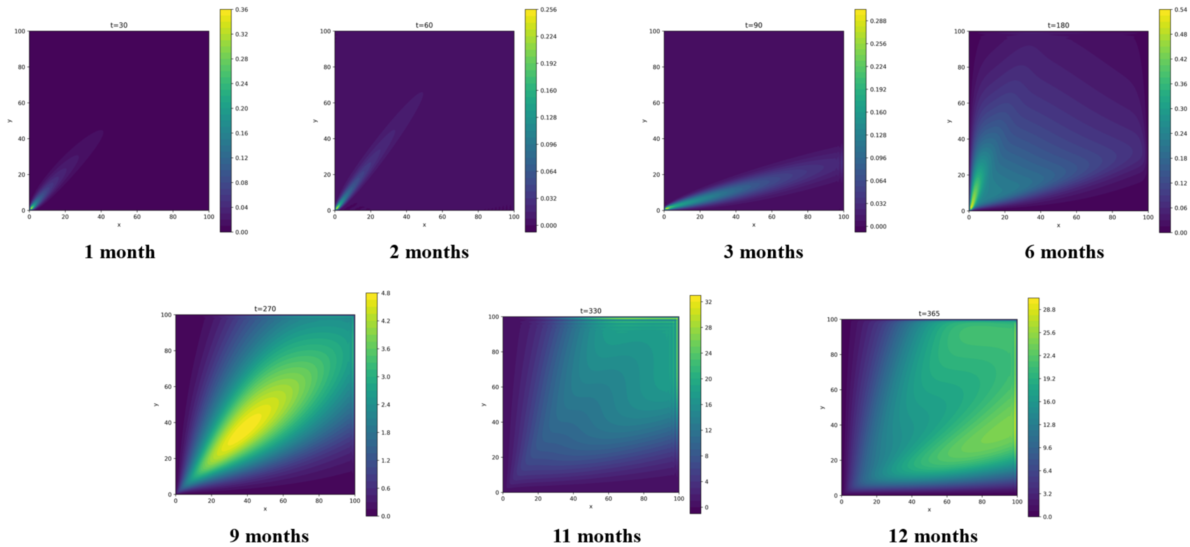


Figure 2: Population Density Distributions of Dandelion Drifting Seeds Over Time

Figure 2 illustrates how the population density of dandelion drifting seeds varies over 1, 2, 3, 6, 9, 11, and 12 months. Because of the initially adjacent dandelion puffball, drifting seeds were produced in the air and flowed across the territory. Based on the effects of advection caused by the random wind and Brownian random motion, drifting seeds produced by the puffball would diffuse freely and probably land on dirt due to gravity. Then the seed develops and the loop continues. From the seven graphs, we can clearly see how the population density of drifting seeds increases, first near the initial puffball's position, then radially spreading out across the territory since each puffball would take seeds farther, and after one year a large area of the land was covered in green colors. Moreover, the overall increase of the units of the color bars in each graph also proves that seeds are rapidly expanding so that the density of seeds flowing in the wind outweighs the density of seeds falling down because of gravity. Consequently, this satisfies our PDE from the DSM and is considered a reasonable outcome of the spread of dandelions.

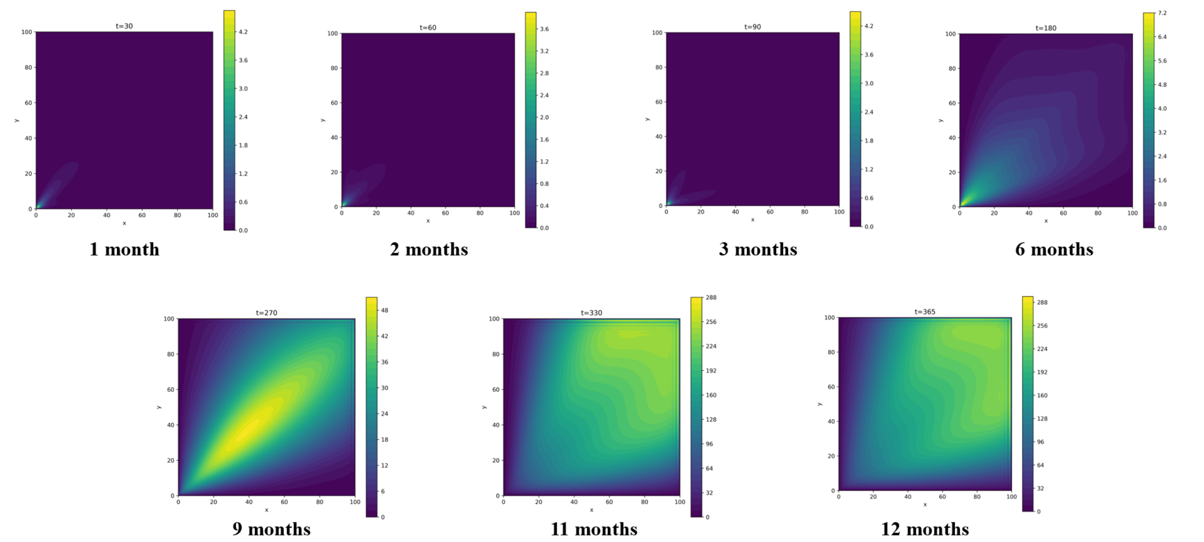


Figure 3: Population Density Distributions of Dandelion Settled Seeds Over Time

Next, Figure 3 identifies how settled seeds distribute over the same periods of time in the given plane reflecting the one-hectare land. Because settled seeds cannot move, their positions mainly depend on tracks of drifting seeds. Comparing the two figures, we can see that the increase in the density of settled seeds is later in time than the increase in the density of drifting seeds. It is worth mentioning that the units are also increasing (so do the latter two graphs). As the overall drifting seed density is evidently increasing due to its spreading, the overall settled seed density also increases dramatically over one year. In fact, while comparing the units of the color bars of the two figures, we can learn that settled seed populations experience a way more dramatic "burst" than diffused seed populations. This is because "drifting" is a much shorter state than "being settled in the dirt" – many drifting seeds fall within a short time period and become settled seeds, so that the population density increases dramatically compared to drifting seeds'.

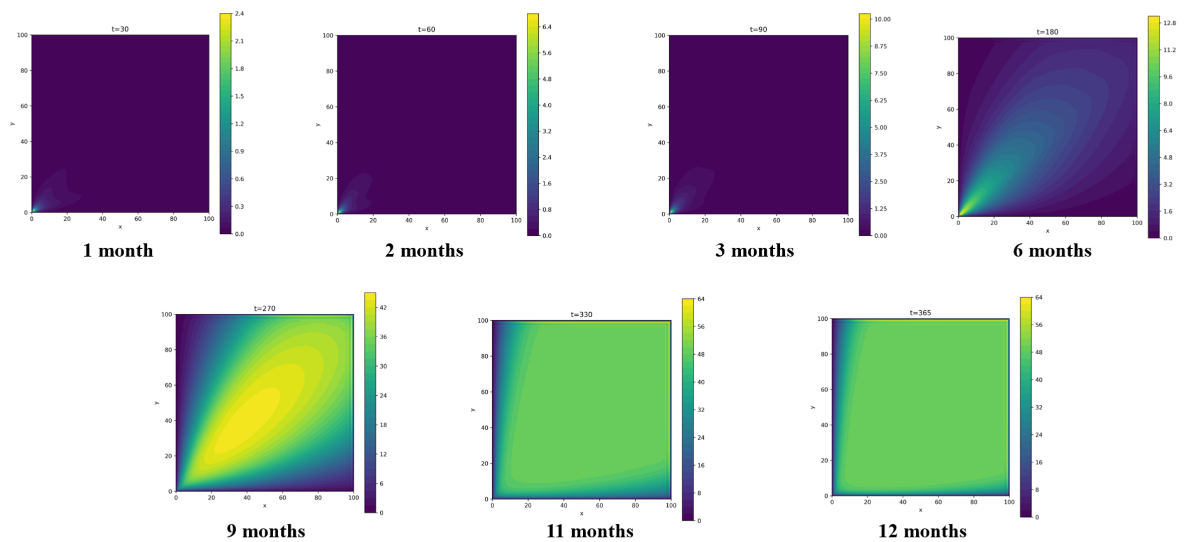


Figure 4: Population Density Distributions of Dandelion Plants Over Time

Correspondingly, the population density of dandelion plants is increasing, as shown in

Figure 4. As the settled seeds germinate, dandelion plants also radially spread wide across nearly the whole region after one year, and because a germinated dandelion does not move, the distribution trend of Figures 3 and 4 are alike. This clearly showcases and predicts how dandelion plants spread over the course of several months under given conditions and using adjusted parameters.

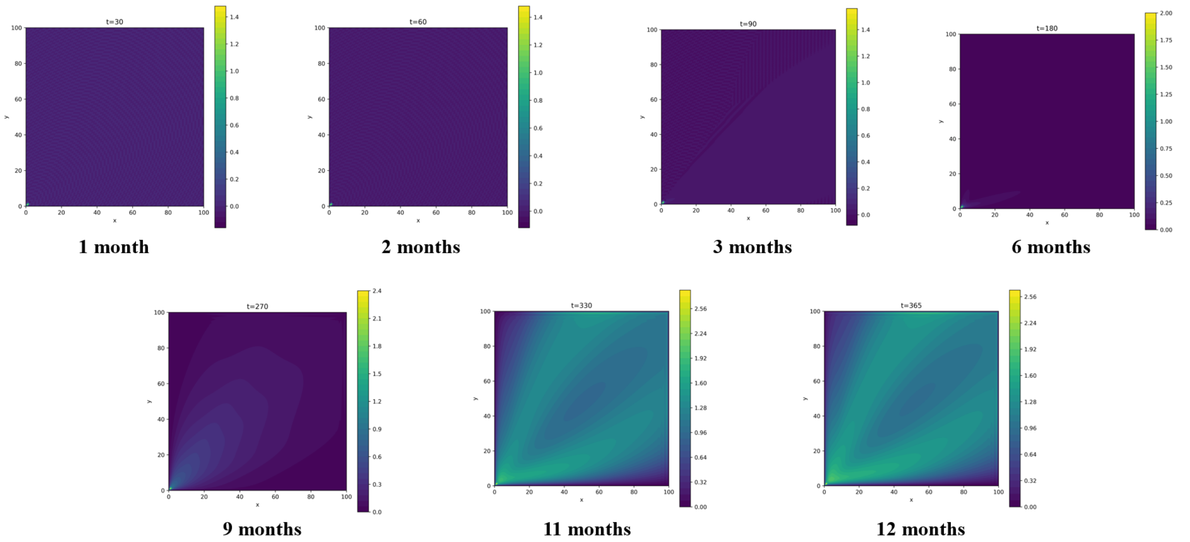


Figure 5: Population Density Distributions of Dandelion Puffballs Over Time

Last but not the least, the distribution of the puffball population's density is increasing but is relatively slower than the previous spread of dandelions. From the slower increase in units of the color bar, we can learn that the "burst" of the population of puffballs is also weaker than previous ones. A possible interpretation is that it takes more time and risks for a dandelion to enter the puffball stage, considering factors such as the death rate so that puffballs certainly have a smaller population. Overall, the puffball still follows the increasing trend of the spread of dandelions. As a result, we can conclude that our DSM model is valid and efficient enough to solve dandelion distribution issues using modeling and visualizations.

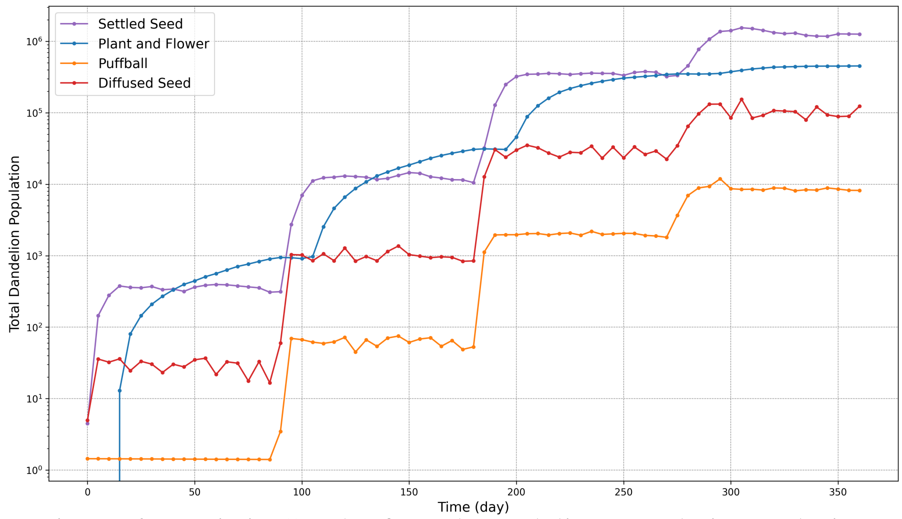
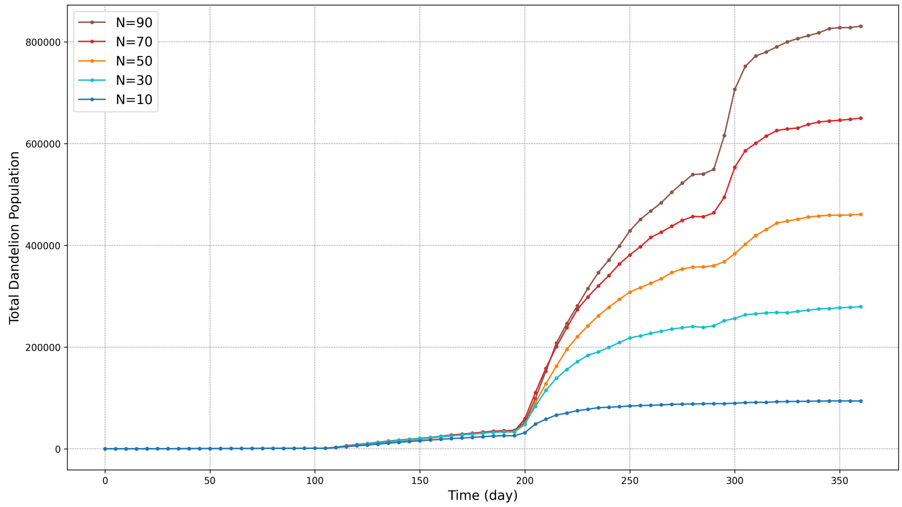


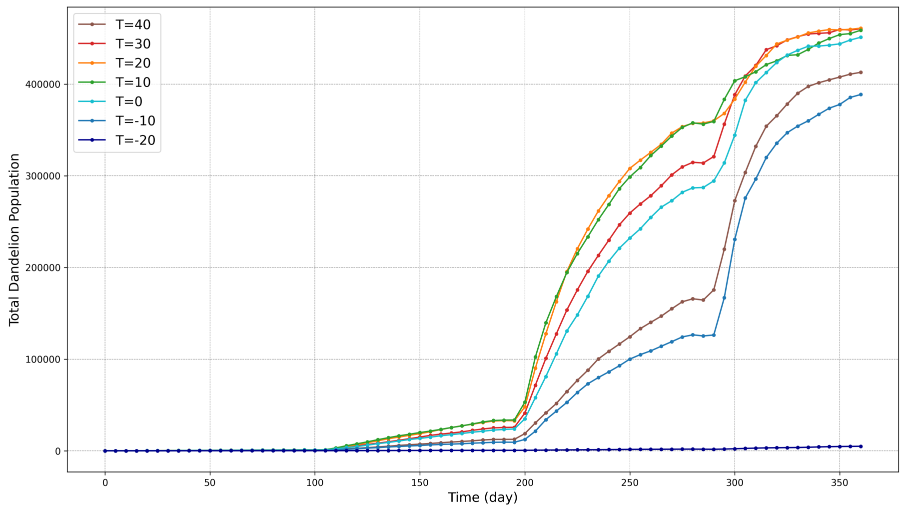
Figure 6: Logistic Graph of Total Dandelion Population and Time

Figure 6 is a modified logistic graph illustrating the relationship between the total population of the four dandelion stages and the time, from zero to a year. It provides further support for our previous investigations as all four dandelion stages experience gradual increases and bursts in populations (note that the y-axis increases exponentially). It is worth mentioning that according to the graph, both the puffball population and diffused seed population are evidently smaller than the settled seed population and the dandelion plant population, corresponding to our observations and interpretations in previous paragraphs of this section.

4.3.3 Sensitivity Analysis

Because the plant stage serves as the most representative and major dandelion life stage, in this section we will conduct a sensitivity analysis that investigates how different variables trigger changes in the growth tendency of the dandelion plant population. Details are shown in the following Figure 7.





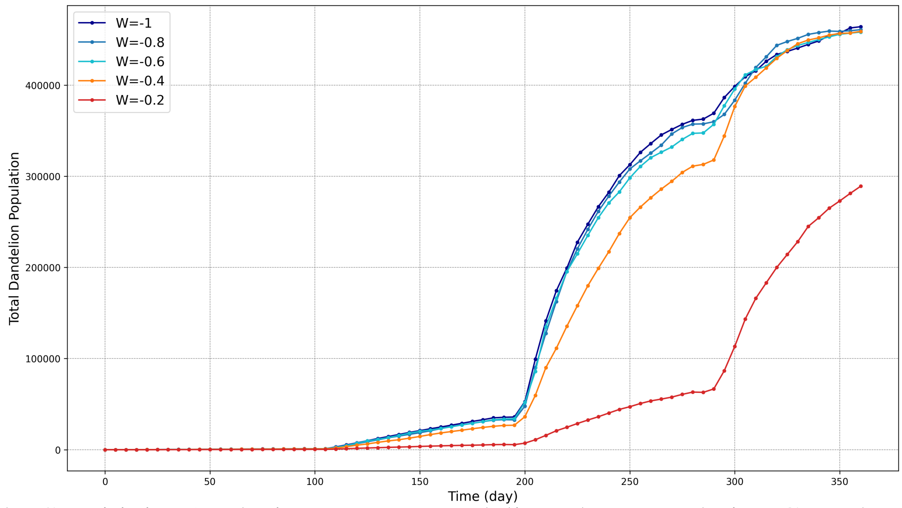


Figure 7: The Sensitivity Analysis Between Dendalian Plant Population Growth Tendency and Different Variables

Based on the results in Figure 7, we can draw multiple conclusions regarding the dandelion plant population's sensitivity. For the first graph, it is clear that the growth rate increases as the maximum density N increases from 10 to 90. It is easy to understand that maximum density restricts population growth because it serves as a density ceiling for dandelions. The population remains low when maximum density is restricted at 10 and increases correspondingly as N becomes more, and grows the fastest when N is at its maximum value 90.

The second graph illustrates the sensitivity according to changing temperatures. We can see that when temperature is between 0 and 30 Celsius, typically some number between 10 to 20 Celsius, the population growth reaches its optimum state and the growth rate reaches its maximum. When T is relatively low, such as -20 Celsius, the population nearly stops growing. Conversely, when T is relatively higher than room temperature, such as 40 Celsius, the population curve also remains lower than curves with intermediate temperatures. This highlights the existence of an optimum temperature for dandelions to grow efficiently.

The last graph focuses on the impact of water potential, or humidity, on the growth rate of the dandelion plant population. It is fair to conclude that as water potential decreases the growth rate increases, but at a decreasing rate – as W approaches -1 the curves become closer to each other.

5 Quantifying Invasiveness of Invasive Species

5.1 Model Correction: Dandelion Spread Under Intraspecific Competitions (DSM-C)

The entry of invasive species into the habitat may lead to competition between native species and the invasive species. The growth of the invasive species can inhibit the growth of native species. Our DSM model in the previous section does not incorporate the effects of interspecific competition on the population or spread of dandelions. In order to solve the task of quantifying the invasiveness of invasive species, we must consider interspecific competitions in our model. Hence, we will propose the modified DSM Model, DSM-C Model, in this section.

Based on literature reading, in a Lotka-Volterra predator-prey model with diffusion, the products of interspecific coefficient α_{ij} and population density of other species u_j are deducted from the traditional logistic population growth model in order to model plants' population density under both interspecific and intraspecific competitions[7].

We adopt the same method while modifying a little regarding the place to deduct the products. Since our DSM model models the population density for different dandelion life stages, to avoid redundancy in calculations, we only change PDE for dandelion plants since plants (in our model plants and flowers have been merged together and thus "plants" refer to both of them) constitute the majority of the dandelion population and are also the ones that require the most resources.

Adding correction terms to the formula, PDE for dandelion plants is now

$$\frac{\partial u_p}{\partial t} = \sigma(n_s) - (1-\delta_p)\cdot\sigma(n_p) - \delta_p \cdot u_p - \sum_{allj} \alpha_{ij} u_j \tag{5.1}$$

with other formulas unchanged. Here u_j represents the population densities of other species. Note that throughout the entire paper, i is always the index of the species the equation is modeling, which is dandelion in this case, and j represents the indices of other species in the region. α_{ij} measures the interspecific effect of j on i's population.

5.2 Modified Logistic Population Growth Model for Other Plant Species Under Competitions

For all the other plants that we want to model, we apply the traditional logistic population growth model with the correction term of interspecific effects. There are three reasons that we do not apply the same DSM-C Model for other plants: 1. We do not have to model the population of every life stages for those plants; 2. The advection-diffusion equation for modeling seed dispersion in the air is not that useful because the spread of seeds for those plants are much less dependent on the wind; 3. To reduce the computational load of the model, which is important when we have coupled PDEs.

The equation to calculate the population density of other plant species is

$$\frac{\partial u_i}{\partial t} = r_i u_i(1 - \frac{u_i}{N_i}) + D_i(\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2}) - \sum_{allj} \alpha_{ij} u_j, \tag{5.2}$$

modified from the Fisher model in the literature[7]. r_i is the growth rate, N_i is the maximum density, and D_i is the diffusion coefficient.

This equation will first be used to model the population densities of native species in the area after dandelion is introduced for Problem 2a. And then it will be used to model the population densities of the two other invasive species as well as the native species for Problem 2b.

5.3 Computing Indicators

For the computation of the "impact factor" which measures the extent of harm that the invasive species brings to a specific region, we have to first determine the indicators. Through careful selections, we decides to incorporate three indicators into the computation of the "impact factor":

1. **Change in total biomass of native species Δw_T :** This reflects on how the invasive

species affects the overall native plantation in the region. The total biomass is a good indicator of the total plantation yield in the area, and can also imply the macro economic value of the whole area.

2. **Economic value Θ**: This can be used to measure the economic value of this invasive plant species, with considerations for its nutritional value, diversity of uses, and its total biomass after invasion.

3. **Impacts on the environment Φ**: This is used to measure how the plant affects the habitat (more specifically, water and nutrition in the soil) by considering both the water and nutrient uptake efficiency of the plant species.

5.3.1 Change in Total Biomass: Modified Yield and Density Equation

The simple equation to calculate a single plant's biomass w obtained from the yield and density equation is

$$w = w_m(1 + au)^{-1}, \tag{5.3}$$

where w_m is the maximum potential biomass per plant and a is the area necessary to achieve w_m[10]. u is still the population density.

The effects of interspecific competition on biomass can be modeled by simply inserting α_{ij}, thus

$$w_i = w_m^i[1 + a_i(\sum_{allj} \alpha_{ij}u_j + u_i)]^{-1}. \tag{5.4}$$

For biomass of dandelion , we substitute u_i by u_p for the same reason with only changing PDE for dandelion plants when considering the effects of interspecific competition on dandelion's population (more details in section 5.1).

We now have the formula to calculate the biomass for species i under competitions. We can calculate the total biomass w_T^i of species i on day t, which is

$$w_T^i = \sum_{x,y=1}^{x,y=100} w_i(x, y, t) \cdot u_i(x, y, t). \tag{5.5}$$

This equation means that we divide the one-hectare plot of land into one hundred squares with areas of 1 m^2, and calculate the total biomass for each square by multiplying the average biomass in the square by the number of plants in the square. We assume the population in a square is evenly distributed, so w_i is equal to the average biomass of plants in the square.

We are able to compute the final value of our Biomass Indicator which is measured by the change in total biomass Δw_T^{native} of native species in the area after one year.

Assume there are n native species, and they are sequentially assigned indices from 1 to n.

$$\Delta w_T^{native} = \sum_{i=1}^{n} \sum_{1 \le x,y \le 100} w_i(x, y, 365) \cdot u_i(x, y, 365) - w_0. \tag{5.6}$$

Note that w_0 is the total mass of all native plant species in the area if the invasive species has never invaded, which means the index of the invasive species is removed from j in equation 5.4 and equation 5.2 when doing calculations for all native species.

5.3.2 Economic Value

The intrinsic economic value of an invasive plant species is determined by three factors: its nutrient density score NDI, which evaluates how much of the essential nutrients does the plant contain; the number of its economic usages η; and its total biomass on day 365 of invasion. There are 12 economic use categories in total, including medicines, materials, environmental, human food, etc[11].

$$\Theta = \eta \cdot NDI \cdot w_T^{invade}. \tag{5.7}$$

5.3.3 Impacts on the Environment

For the invasive species' impacts on the environment, we mainly focus on two aspects: its water uptake efficiency, reflected by soil's interrill soil erodibility K_i with the plant planted[12]; and its nutrient uptake efficiency reflected by its bioconcentration factor BCF. Multiplying the two variables together, we get

$$\Phi = K_i \cdot BCF, \tag{5.8}$$

where $K_i = 3.55e^{-0.71RD}$; RD is the plant species' dead root mass[444].

5.4 Generating the Weight of Impact Factor: Analytic Hierarchy Process (AHP)

We use AHP method to generate the final "impact factor". Let's first construct a judgment matrix to compare the importance of indicators.

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 1/5 & 1 & 2 \\ 1/4 & 1/2 & 1 \end{bmatrix}$$

Δw_T^{native}, Θ and Φ are denoted by 1, 2, 3, respectively, and the value in the i-th row and j-th column represents the importance of the i-th indicator relative to the j-th indicator. A value of 1 indicates equal importance, and the larger the number, the more important the i-th indicator is compared to the j-th indicator.

According to AHP, the variable weight is calculated using the eigenvector corresponds to the largest eigenvalue of matrix A. The eigenvector calculated is:

$$v = \begin{bmatrix} 2.03 \\ 0.57 \\ 0.39 \end{bmatrix} \tag{5.9}$$

Normalizing the eigenvector, we obtain that the weight of the three indicators are respectively:

Hence the "Impact Factor" (I) is

$$I = 0.68\Delta_T^{native} + 0.19\Theta + 0.13\Phi. \tag{5.10}$$

Variable	Δw_T^{native}	Θ	Φ
Weight	0.68	0.19	0.13

Table 5.1: Indicators Weight

5.5 Initial Settings

To apply the model, we have to first simulate the environment for the invasive species to enter. We will still use the one hectare plot of land to generate the "Impact Factor". We select two native species to live on this land: wheat and annual ryegrass.

Based on data obtained in past researches[10, 13, 14, 15], and through careful empirical analysis, we set the initial parameters for the two native species as follow.

Plant	Initial Population	r_i	N_i	w_m	a_i	D_i
Wheat	250000	0.2	30	100	0.61	5
Ryegrass	450000	0.1	15	100	0.29	10

Table 5.2: Initial Parameter Settings

The initial population densities of the two species is derived by assuming that the initial population is uniformly distributed across the land, so the initial population density for every spot on the land is equal to initial population/10000 m^2.

5.6 Applying the Model for Dandelion

We are using the same data for the parameters from problem 1, and we determined other parameters based on relevant literature readings [10, 12, 16, 17, 18], and made careful estimations and empirical analysis.

Plant	w_m	a_i	N_i	D_i	ψ	n_s	n_p	n_{puf}	λ	g_{base}	δ_p^{base}	δ_{puf}^{base}
Dandelion	100	0.61	50	10	0.9	14	90	95	20	0.5	5.48×10^{-4}	2.74×10^{-4}

Table 5.3: Initial Parameter Settings for Dandelion(a)

Plant	η	NDI	RD	BCF
Dandelion	3	46.34	0.092	<1(0.6)

Table 5.4: Initial Parameter Settings for Dandelion (b)

Estimations of interspecific coefficient are made carefully through data fitting based on species' shoot: root ratios.

$$\alpha_{ij}^{dande} = \begin{bmatrix} 0 & 0.3125 & 1.1 \\ 3.2 & 0 & 0 \\ 0.909 & 0 & 0 \end{bmatrix}$$

The value in the i-th row and j-th column represents the interspecific effect of species j on species i. Dandelion, wheat, and ryegrass are respectively indexed as 1, 2, 3. Since we are modeling the effects of invasion, we assume the native species' interspecifc competition coefficients are zero.

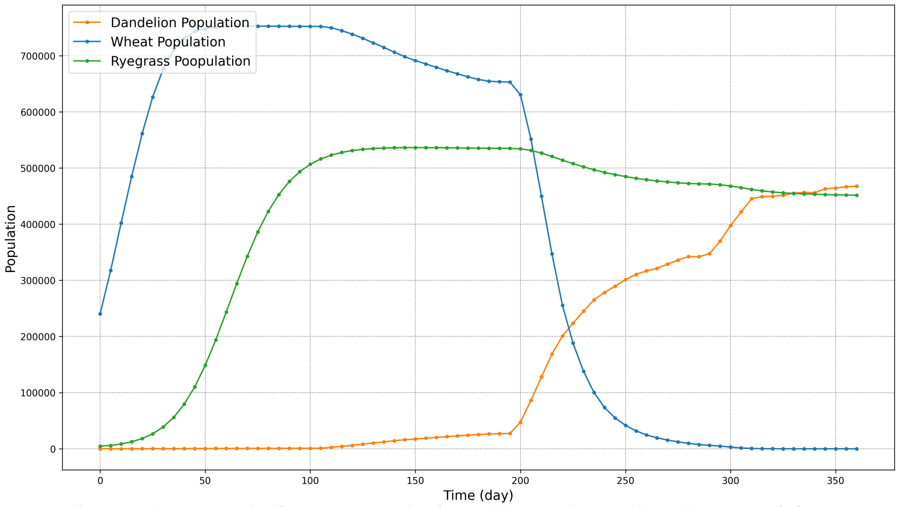


Figure 8: Dandelion: Population Growth Under Competitions

Figure 8 shows the change of population for the three species overtime. We can see that initially the growth rates of wheat and ryegrass are in a stable phase; however, as the dandelion population increases, the wheat population significantly declines, while the fluctuations in the ryegrass population are not as noticeable. This is because the impact of dandelions on wheat is quite significant, whereas their effect on ryegrass is less pronounced.

Variable	Δw_T^{native}	Θ	Φ
Unnormalized Value	-769856	215243×10^3	1.995
Normalized Value	-1	1	-0.9929

Table 5.5: Calculated indicator value

We obtain that $I_{dandelion} = -0.61907$.

5.7 Appyling the Model for Millet and Cereal

The data are obtained from relevent papers [11, 12, 19, 20, 21, 22] and also through very careful empirical analysis. Again, estimations of interspecific coefficient are made carefully through data fitting based on species' root: shoot ratios.

Plant	r_i	N_i	w_m	a_i	D_i	η	NDI	RD	BCF
Millet	0.2	15	180	0.61	5	3	15	0.17	1.3
Cereal	0.2	15	230	0.29	10	4	16	0.13	0.3

Table 5.6: Initial Parameter Settings

$$\alpha_{ij}^{millet} = \begin{bmatrix} 0 & 0.23 & 0.14 \\ 4.26 & 0 & 0 \\ 7.12 & 0 & 0 \end{bmatrix}; \alpha_{ij}^{cereal} = \begin{bmatrix} 0 & 0.75 & 0.18 \\ 0.66 & 0 & 0 \\ 5.31 & 0 & 0 \end{bmatrix}. \tag{5.11}$$

The value in the i-th row and j-th column represents the interspecific effect of species j on species i. Millet, wheat, and ryegrass are respectively indexed as 1, 2, 3; and then cereal, wheat, and ryegrass are respectively indexed as 1,2,3.

plant	Variable	Δw_T^{native}	Θ	Φ
Millet	Unnormalized Value	-3.206$\times 10^8$	6.3406$\times 10^9$	4.09
Millet	Normalized Value	-1.0	1.0	-0.9037
Cereal	Unnormalized Value	-9.862 $\times 10^7$	6.316 $\times$ 10^9	0.971
Cereal	Normalized Value	-1	1	-0.9693

Table 5.7: Calculated indicator value

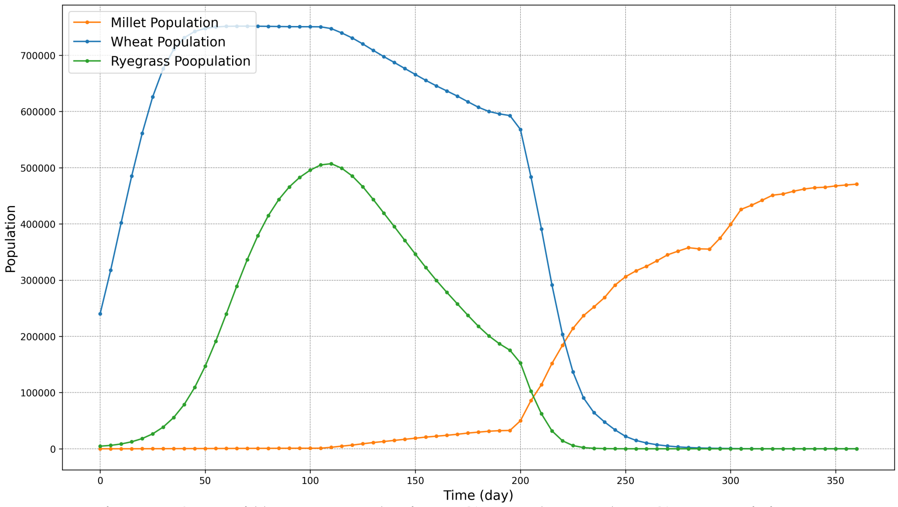


Figure 9: Millet: Population Growth Under Competitions

With the introduction of millet, both wheat and ryegrass populations are significantly affected, resulting in a substantial decrease in their populations.

$I_{millet} = -0.608.$

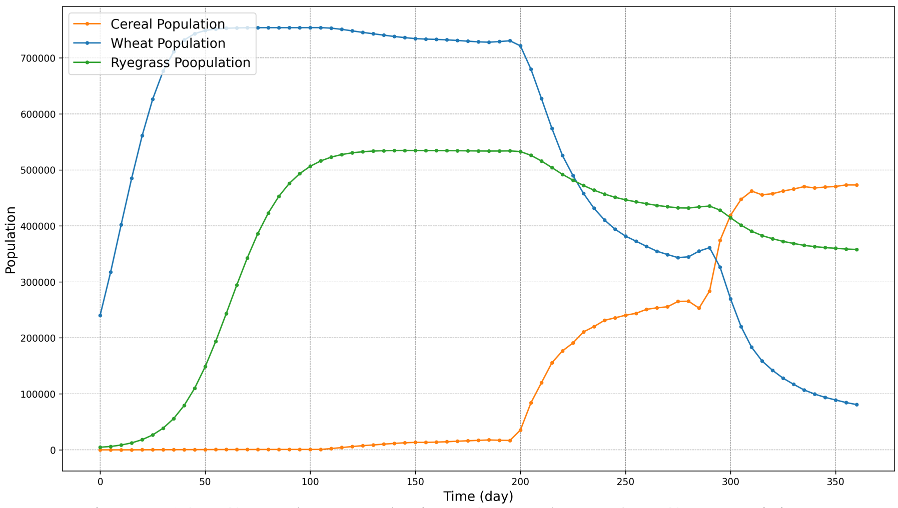


Figure 10: Cereal: Population Growth Under Competitions

With the introduction of cereal, both wheat and ryegrass are observed to have a decreasing trend in their population. However, their rate of decline gradually leveled off towards the end, and ryegrass is less influenced compared to wheat.

$I_{cereal} = -0.616.$

6 Model Evaluation

6.1 Strengths

1. The model incorporates the use of partial differential equation, which is able to capture the location information of dandelions. It also enables the modeling of the wind's influence on the spread of the dandelion seed, which is crucial for plant species which use wind as the main source of seed spread.

2. The model splits the life of dandelion into different stages, which makes it possible to specifically predict the population of dandelion at each stage.

3. The model incorporates the concept of biomass, which can both infer micro-economic value and better estimate the loss caused by the invasion.

6.2 Weaknesses

1. Approximating the solution of PDE system is time-consuming, which makes it inconvenient for the large scale testing of the model.

2. Since the model does not incorporate a limit for the influence of the invasive species, it may over-estimate the spread of invasive species. A solution of this might be applying a logistic function or limitation function to the interspecific competition term.

7 Conclusion

In conclusion, we first constructed the Dandelion Spread PDE model to investigate the spread of dandelion in different life stages. We then purposed the modified model to incorporate the effects of interspecific competition on the population. We purposed another model which is the modified Fisher model to model other plant species. Then, we determined the indicators of the "impact factor" and used AHP to obtain the weight of each indicator. Finally, we obtain the "impact factors" for the three invasive species. $I_{millet} > I_{cereal} > I_{dandelion}$. Dandelion appears to be most invasive.

Overall, our models have good performances and present accurate predictions of the population growth. Our model may be improved in the future by incorporating a limit for the influence of the invasive species to make the results more realistic.

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Appendices

Formula for the regressed equation $K_t(T)$:

$$K_t(T) = -0.000669087136929461 \cdot T^2 + 0.021680497925311227 \cdot T + 0.8253371369294603 \quad (.1)$$

Formula for the regressed equation $K_w(W)$:

$$K_w(W) = \frac{1}{1 + e^(2.5759772066681115 + 11.409865444179479 \cdot W)} \quad (.2)$$

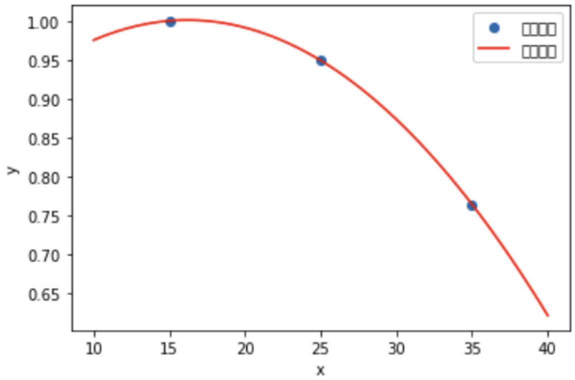

Figure 11: Quadratic Function Regression of $K_t(T)$

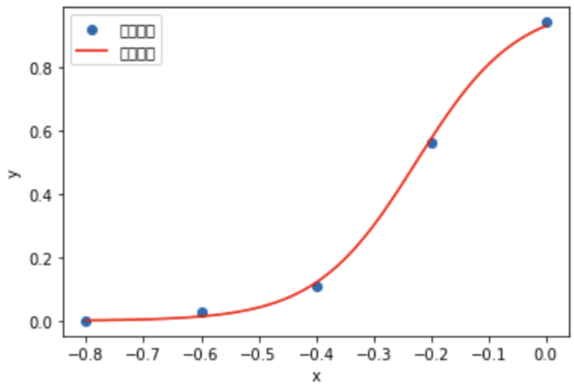

Figure 12: Logistic Function Regression of $K_w(W)$